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LETTER TO THE EDITOR

$N=1$ superconformal invariance in the six-state quantum chain with free boundary conditions

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Abstract. The spectrum of a six-state one-dimensional quantum chain with free boundaries at the critical point is studied. Using finite-size scaling and conformal invariance, we find a line in the plane of the coupling constants where the system has $N=1$ superconformal invariance with $c=\frac{5}{4}$.

The six-state one-dimensional quantum chain is defined by the D_6-symmetric self-dual Hamiltonian (von Gehlen and Rittenberg 1986b, 1987)

$$H=-\frac{1}{\xi}\sum_{i=1}^{N}\{\sigma_i+\sigma_i^5+\varepsilon(\sigma_i^2+\sigma_i^4)+\delta\sigma_i^3 \\ +\lambda[\Gamma_i\Gamma_{i+1}^5+\Gamma_i^5\Gamma_{i+1}+\varepsilon(\Gamma_i^2\Gamma_{i+1}^4+\Gamma_i^4\Gamma_{i+1}^2)+\delta\Gamma_i^3\Gamma_{i+1}^3]\} \qquad (1)$$

where the 6×6 matrices σ and Γ are

$$\sigma=\omega^{i-1}\delta_{i,j} \qquad \Gamma=\delta_{i-1,j} \qquad (2)$$

with $\omega=\exp(2\pi i/6)$ and $\delta_{0,j}=\delta_{6,j}$, $i,j=1,\ldots,6$, where N represents the number of sites, λ plays the role of the inverse temperature and ε and δ are coupling constants determining the global symmetry of the Hamiltonian. For special values of the couplings the symmetry is higher than D_6 (Badke *et al* 1985), $\varepsilon=\delta=1$ corresponds to the six-state Potts model. In this letter we consider only the ferromagnetic region of interaction (denoted by I in figure 1) in the critical plane $\lambda=1$. It is defined by

$$\varepsilon>-1 \qquad \varepsilon>-\tfrac{1}{3}(1+2\delta) \qquad \delta>-2. \qquad (3)$$

For free boundary conditions we have $\Gamma_{N+1}=0$. The normalisation ξ, which fixes the Euclidean timescale, can be obtained numerically (von Gehlen *et al* 1986).

Part of the phase diagram where the system has been studied is shown in figure 1. von Gehlen and Rittenberg (1986b, 1987) found numerically for three special values of the coupling constants ($\delta=1$, $\varepsilon_1=0$, $\varepsilon_2=\frac{1}{3}$, $\varepsilon_3=\frac{3}{5}$) a central charge $c=\frac{5}{4}$ of the Virasoro algebra and for $\varepsilon=0$ anomalous dimensions (=critical exponents) which are close to those expected from $N=1$ supersymmetry. On the other hand Zamolodchikov and Fateev (1985) suggested, that for $\varepsilon=\sqrt{\frac{1}{3}}$, $\delta=\frac{1}{2}$ the model should exhibit $N=1$ superconformal invariance with $c=\frac{5}{4}$. Alcaraz (1987) found the central charge predicted by Zamolodchikov and Fateev and a few of the exponents measured by von Gehlen and Rittenberg for $\varepsilon=0$, $\delta=1$. Now a more detailed study of the model in the ferromagnetic domain completes the partial information described above. A region with central charge $c=1.25$, marked by A, was found, including the points at which

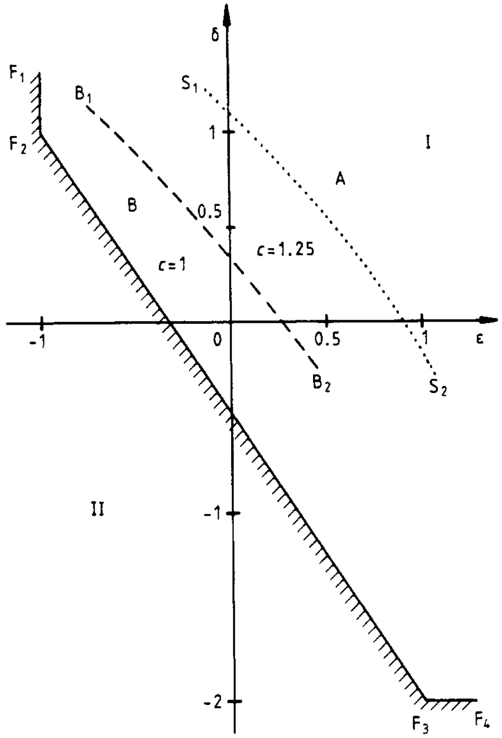

Figure 1. Parameter plane of the Hamiltonian (1). Line $F_1F_2F_3F_4$ marks the border between the ferromagnetic (I) and antiferromagnetic (II) domain of interactions. A region of central charge $c = 1.25$ is marked by A. Along the dotted line S_1S_2 we find $N = 1$ superconformal invariance. Region B, bounded by B_1B_2, shows $c = 1$.

von Gehlen and Rittenberg (1986b, 1987) and Alcaraz (1987) made measurements. This region contains a curve of $N=1$ supersymmetry (curve S_1S_2). Furthermore, numerical data which will be presented elsewhere show a second region, denoted B, with $c=1$. The boundary of B is given by curve B_1B_2. Whether c jumps from $c=1$ to $c=1.25$ or varies continuously has to be clarified. In this letter the finite-size scaling limit of the spectrum of H with free boundaries for three values of couplings (see table 1 in which the corresponding values of ξ are also listed) on the supersymmetric line S_1S_2 will be given. A paper presenting the full operator content for all boundary conditions will be published elsewhere.

The finite-size scaling limit of the spectrum for free boundaries is defined by the quantities (Cardy 1984, 1986, von Gehlen and Rittenberg 1986a):

$$\mathscr{E}_k(r) = \lim_{N\to\infty} \frac{N}{\pi}(E_k(r) - E_F) \tag{4}$$

Table 1. Normalisation factor ξ and central charge c for three couplings of ε, δ.

ε, δ	0.1, 1.0	$1/\sqrt{3}$, 0.5	0.9, 0.0
ξ	2.51 (1)	3.002 (2)	3.050 (3)
c	1.24 (2)	1.25 (1)	1.25 (2)

where $E_k(r)$ is a certain energy level of H for N sites and E_F denotes the ground-state energy, i.e. the lowest eigenvalue of H. As a consequence of conformal invariance in two dimensions these quantities are given by the unitary representations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}c(n^3-n)\delta_{n+m,0} \qquad n, m \in Z. \tag{5}$$

An irreducible representation is characterised by the highest weight

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \qquad L_n|\Delta\rangle = 0 \qquad n>0 \tag{6}$$

where Δ is a surface critical exponent and its conformal tower gives a contribution to the spectrum

$$\mathscr{E}(r) = \Delta + r \qquad r = 0, 1, 2, \ldots \tag{7}$$

with degeneracy $D(\Delta, r)$. The levels $|\Delta + r\rangle$ have a relative parity $(-1)^r$ to $|\Delta\rangle$.

The Virasoro algebra may be extended to a $N=1$ superconformal algebra (Friedan *et al* 1985, Berdshadski *et al* 1985, Eichenherr 1985) given by

$$[L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r} \tag{8}$$

$$\{G_r, G_s\} = 2L_{r+s} + \tfrac{1}{3}c(r^2 - \tfrac{1}{4})\delta_{r+s,0} \tag{9}$$

together with (5), where $r, s \in Z$ in the Ramond sector and $r, s \in Z + \frac{1}{2}$ in the Neveu-Schwartz sector. For $c < 1.5$ the central charge and the anomalous dimensions are quantised:

$$c = \tfrac{3}{2} - 12/m(m+2) \qquad m \geqslant 2 \tag{10a}$$

$$\Delta_{p,q} = \frac{[p(m+2)-qm]^2 - 4}{8m(m+2)} + \frac{1-(-1)^{p-q}}{32} \tag{10b}$$

where $1 \leqslant p < m$, $1 \leqslant q < m+2$, $p-q$ is even in the Neveu-Schwartz sector and $p-q$ is odd in the Ramond sector. For $c = \frac{5}{4}$ we have $m = 6$.

In order to compute the spectrum of the Hamiltonian (1), we use the D_6 symmetry to prediagonalise H into block matrices. The dihedral group D_6 is of order twelve corresponding to the transformations

$$(\Gamma_i')^m = A^{mn}(\Gamma_i)^n \qquad m, n = 1, \ldots, 5 \tag{11}$$

$$A = \Sigma^l C^k = \omega^{ml}\delta_{m,n}(\delta_{6-m,n})^k \qquad l = 0, \ldots, 5; \quad k = 0, 1. \tag{12}$$

The Σ^l build the cyclic group Z_6 and C is the charge conjugation matrix. The Z_6 symmetry allows a prediagonalisation into six charge sectors with charge $Q = 0, \ldots, 5$. The corresponding Hamiltonian will be denoted H_Q.

D_6 has two irreducible two-dimensional representations

$$D_q(\Sigma^l) = \begin{pmatrix} \omega^{ql} & 0 \\ 0 & \omega^{-ql} \end{pmatrix} \qquad D_q(C) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad q = 1, 2 \tag{13}$$

which mix the sector 1 with sector 5 and sector 2 with sector 4, so that

$$H_1 = H_5 \qquad H_2 = H_4 \tag{14}$$

and has four one-dimensional representations

$$D_{q,\pm}(\Sigma^l) = (-1)^{ql} \qquad D_{q,\pm}(C) = \pm 1 \qquad q = 0, 3 \tag{15}$$

Table 2. Finite-size scaling limits of the spectrum in the charge-zero sector for positive charge conjugation ($Q, C = 0, +$), for negative charge conjugation ($Q, C = 0, -$), for $Q = 1$ and 2 and for ($Q, C = 3, +$) and ($Q, C = 3, -$). The degeneracies d of the anomalous dimensions $x = \Delta + r$ are predicted from $N = 1$ supersymmetry. The columns on the right contain the measured values for three couplings ε, δ for positive and negative parity ($P = \pm 1$).

					Measured finite-size scaling limits		
Q, C	P	$\Delta + r$	x	$d(\Delta, r)$	$\varepsilon = 0.1, \delta = 1.0$	$\varepsilon = 1/\sqrt{3}, \delta = 0.5$	$\varepsilon = 0.9, \delta = 0.0$
0, +	+	0+2	2.0	1	1.96(1)	2.00(4)	1.97(1)
	−	0+3	3.0	1	2.93(4)	2.96(2)	2.93(3)
	+	0+4	4.0	3	3.6(1) 3.7(1) 3.92(4)	4.0(1) 4.0(2) 3.96(3)	3.8(2) 3.95(5) 3.95(5)
	−	0+5	5.0	3	4.64(4) 4.98(5) 5.00(4)	4.8(1) 5.00(5) 4.93(4)	4.8(2) 4.9(1) 4.95(5)
	+	0+6	6.0	7	5.9(1) . . .	5.8(2) 6.0(1) . . .	5.8(4) . . .
	−	0+7	7.0	8	7.2(2) . . .	7.0(2) 6.8(3) 6.8(3) . . .	
0, −	−	3	3.0	1	2.83(3)	3.00(5)	2.83(3)
	+	3+1	4.0	1	3.7(2)	4.00(5)	3.82(2)
	−	3+2	5.0	3	4.79(5) 4.9(2) 4.8(3)	5.0(1) 4.9(1) 4.95(5)	4.5(3) 4.8(1) 5.2(5)
	+	3+3	6.0	4	5.8(1) 6.0(2) . . .	5.9(2) 6.0(2) 6.0(3) 6.3(3)	5.7(4) 5.9(6) 6.0(2) 6.0(2)
1	+	$\frac{5}{6}$	0.833	1	0.77(1)	0.833(5)	0.76(1)
	−	$\frac{5}{6}+1$	1.833	1	1.76(1)	1.83(1)	1.77(1)
	+	$\frac{5}{6}+2$	2.833	2	2.7(1) 2.78(5)	2.75(4) 2.84(2)	2.71(1) 2.71(1)
	−	$\frac{5}{6}+3$	3.833	3	3.4(1) 3.7(1) 3.8(1)	3.81(5) 3.7(1) 3.8(1)	3.6(1) 3.7(1) 3.7(2)
	+	$\frac{5}{6}+4$	4.833	6	4.6(1) 4.7(1) 4.7(1) 4.7(1) 4.7(2) 4.8(2)	4.6(1) 4.7(2) 4.8(2) 4.7(1) . . .	4.7(2) 4.7(2) 4.9(3) 4.7(2) 4.5(1) 4.7(2)
	−	$\frac{5}{6}+5$	5.833	9	5.5(5) . . .	5.6(2) 5.7(1) 5.7(2) 5.7(1) 5.7(2) 5.7(3) 5.9(3) 6.0(3) . . .	5.8(3) . . .

2	+	$\frac{4}{3}$	1.333	1	1.25(1)	1.333(5)	1.27(5)
	−	$\frac{4}{3}+1$	2.333	1	2.17(5)	2.34(2)	2.2(1)
	+	$\frac{4}{3}+2$	3.333	3	3.1(2) 3.1(1) 3.2(3)	3.3(1) 3.2(1) 3.2(1)	3.2(1) 3.2(2) 3.2(3)
	−	$\frac{4}{3}+3$	4.333	4	3.9(2) 4.0(1) 4.2(3) 4.2(4)	4.1(2) 4.2(2) 4.3(1) 4.2(1)	3.8(2) 4.0(2) 4.1(2) . . .
	+	$\frac{4}{3}+4$	5.333	8	5.1(5) 5.1(5) 5.1(2) . . .	5.4(2) . . .	5.2(5) . . .
	−	$\frac{4}{3}+5$	6.333	11	6.3(2) 6.3(5) . . .	6.0(3) . . .	
3, +	+	$\frac{3}{2}$	1.5	1	1.50(3)	1.499(2)	1.44(3)
	−	$\frac{3}{2}+1$	2.5	1	2.4(1)	2.50(1)	2.38(2)
	+	$\frac{3}{2}+2$	3.5	2	3.3(1) 3.44(4)	3.50(3) 3.52(5)	3.3(2) 3.42(5)
	−	$\frac{3}{2}+3$	4.5	3	4.4(4) 4.4(4) 4.4(4)	4.5(1) 4.4(1) 4.5(3)	4.3(1) 4.45(5) 4.44(5)
	+	$\frac{3}{2}+4$	5.5	5	4.95(5) . . .	5.4(1) 5.4(3) 5.4(2) 5.1(2) 5.5(1)	5.2(5) 5.3(3) 5.3(3) 5.4(1) 5.4(4)
	−	$\frac{3}{2}+5$	6.5	7		6.6(3) . . .	6.6(5) . . .
3, −	+	$\frac{7}{2}$	3.5	1	3.34(5)	3.52(3)	3.3(1)
	−	$\frac{7}{2}+1$	4.5	2	4.3(1) 4.3(2)	4.54(4) 4.52(4)	4.21(5) 4.3(1)
	+	$\frac{7}{2}+2$	5.5	3	5.1(3) 5.2(4) 5.4(4)	5.44(4) 5.3(1) 5.5(1)	5.3(3) 5.3(2) 5.5(5)
	−	$\frac{7}{2}+3$	6.5	5	6.5(3) . . .	6.1(2) 6.3(2) 6.4(3) 6.5(4) 6.7(2)	6.5(3) . . .
	+	$\frac{7}{2}+4$	7.5	9		7.4(3) . . .	7.5(5) . . .

which split the charge zero and charge three sectors into two subsectors $0,C$, $3,C$ with eigenvalues of charge conjugation being $C=\pm 1$. Since the Hamiltonian is also parity invariant ($P=\pm 1$), the following twelve matrices $H^P_{Q,C}$ have to be diagonalised for N sites:

$$H^{\pm}_{0,\pm} \qquad H^{\pm}_{3,\pm} \qquad H^{\pm}_{1} \qquad H^{\pm}_{2}. \tag{16}$$

The lower part of the spectra of the twelve matrices was computed and the limits (4) calculated applying the algorithm of Burlisch and Stoer (1964). We obtain the following results:

(i) The central charge c of the Virasoro algebra (5), which can be obtained from the finite-size scaling corrections to the ground-state energy E_F (4) (Blöte *et al* 1986, Affleck 1986) is $c=\frac{5}{4}$ ($m=6$ in (10*a*)) along the line S_1S_2 (table 1).

(ii) Table 2 shows the finite-size scaling limits for the three couplings. The columns on the left give the prediction of the $N=1$ superconformal algebra ($c=\frac{5}{4}$). The columns on the right contain the 'measured' values for positive and negative parity with the estimated errors. Notice that not only the critical exponents are those expected, but also the degenaracies $d(\Delta, r)$ which have been computed from the character expressions of Goddard *et al* (1986). Combining sectors, we see that the sums are given by:

$$\mathscr{E}_{0,+}+\mathscr{E}_{3,+} \qquad (\Delta_{1,1})^{NS}=(0)^{NS} \tag{17a}$$

$$\mathscr{E}_{0,-}+\mathscr{E}_{3,-} \qquad (\Delta_{5,1})^{NS}=(3)^{NS} \tag{17b}$$

$$\mathscr{E}_{1}+\mathscr{E}_{2}=\mathscr{E}_{5}+\mathscr{E}_{4} \qquad (\Delta_{3,1})^{NS}=(\tfrac{5}{6})^{NS} \tag{17c}$$

with

$$\mathscr{E}_{Q,C}=\mathscr{E}^{+}_{Q,C}+\mathscr{E}^{-}_{Q,C} \qquad \mathscr{E}_{Q}=\mathscr{E}^{+}_{Q}+\mathscr{E}^{-}_{Q}. \tag{18}$$

Here $(\Delta_{p,q})^{NS}$ denotes the irreducible representation with highest weight $\Delta_{p,q}$ (10*b*) in the Neveu-Schwartz sector.

Summing up, we find that the central charge and the critical exponents are independent from the coupling constants along the line S_1S_2 in figure 1. The spectrum is given by the representations $(0)^{NS}$, $(3)^{NS}$ and $2\times(\frac{5}{6})^{NS}$ of the $N=1$ superconformal algebra with $c=\frac{5}{4}$. The corresponding operators build a closed subalgebra.

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